LECTURE NOTES
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1 Complementarity Methods

Some problems take the form of complementarity problems as opposed to root-finding or fixed-point problems. The problem is to find an n-dimensional vector $\mathbf{x} \in [\mathbf{a}, \mathbf{b}]$ that satisfies

$$\begin{aligned} x_i > a_i \quad &\Rightarrow \quad f_i(x) \geq 0 \quad \forall i = 1, \ldots, n \\ x_i < b_i \quad &\Rightarrow \quad f_i(x) \leq 0 \quad \forall i = 1, \ldots, n \end{aligned}$$

where a_i and b_i are the ith elements of the n-dimensional vectors a and b and $\mathbf{f}$ maps $\mathbb{R}^n$ to $\mathbb{R}^n$. The condition says that $f_i(x) = 0$ whenever $a_i < x_i < b_i$. It thus includes the root-finding problem as the special case where $a_i = -\infty$ and $b_i = \infty$ for all i. The complementarity problem, however, is not to find a root that lies within specified bounds. An element of $f_i(x)$ may be nonzero at a solution of a complementarity problem but only if x_i equals one of its bounds.

It is true that $\mathbf{x}$ solves the complementarity problem if and only if it solves the root-finding problem given by

$$\tilde{\mathbf{f}}(\mathbf{x}) = \min(\max(\mathbf{f}(\mathbf{x}), \mathbf{a} - \mathbf{x}), \mathbf{b} - \mathbf{x}) = \mathbf{0}$$

where $\min$ and $\max$ are taken row-wise. This can be seen with graphs or proven by looking at cases.

Having reformulated the complimentary problem as a root-finding problem, we can solve it using standard root-finding algorithms. To implement Newton's method although, we will need the Jacobian $\tilde{J}$ of $\tilde{f}$. The ith row of $\tilde{J}$ can be written as

$$\tilde{J}_i(x) = \begin{cases} J_i(x), & a_i - x_i < f_i(x) < b_i - x_i, \\ -I_i, & \text{otherwise}, \end{cases}$$

where J_i is the ith row of J the Jacobian of f and I_i is the ith row of the identity matrix.

While this approach often works, despite the kinks in $\tilde{f}$, other times the kinks may cause Newton's method to fail. An alternative approach is to replace $\tilde{f}$ with a function that has the same roots but is smoother and therefore less prone to numerical difficulties. One function that seems to work is Fischer's function,

$$\hat{f}(x) = \phi^{-}(\phi^{+}(f(x), a - x), b - x),$$

where

$$\phi_i^{\pm}(u, v) = u_i + v_i \pm \sqrt{u_i^2 + v_i^2}.$$

The semismooth formulation is more robust than the minmax formulation but also requires more flops per an iteration.

One example of a place where complementarity problems arise are in constrained optimization problems. Finding an x that satisfies the Karush-Kuhn-Tucker conditions for a maximization problem with the simple bound constraints that $\mathrm{x} \in [\mathrm{a}, \mathrm{b}]$ is equivalent to solving the complementarity problem. For example consider the following static problem

$$\max_{c,h} \frac{c^{1-\sigma}}{1-\sigma} - \alpha \frac{h^{1+\gamma}}{1+\gamma}$$

subject to

$$c = wh + x,$$

$$0 \leq h \leq 1,$$

$$c \geq 0.$$

The Karush-Kuhn-Tucker conditions are

$$(wh + x)^{-\sigma} w - \alpha h^{\gamma} + \lambda - \delta = 0$$

$$h \geq 0, \qquad \lambda \geq 0, \qquad \lambda h = 0$$

$$h \leq 1, \qquad \delta \geq 0, \qquad \delta(h - 1) = 0$$

An $h \in [0, 1]$ that satisfies these conditions will also satisfy

$$h > 0 \Rightarrow (wh + x)^{-\sigma} w - \alpha h^{\gamma} \geq 0$$

$$h < 1 \Rightarrow (wh + x)^{-\sigma} w - \alpha h^{\gamma} \leq 0$$

Finding an h that satisfies the complimentary problem is equivalent to finding an h that solves

$$\min(\max(w(wh+x)^{-\sigma} - \alpha h^{\gamma}, -h), 1-h) = 0.$$

An alternative approach for this problem is to notice that the inequality constraint $h \geq 0$ is never binding. So $h = 0$ will never be a solution. In addition, negative values of h can potentially create problems. For example, if $0 < \gamma < 1$ then h^{γ} is not a real number. Unfortunately the nonlinear equation solver doesn't know this and may try to guess negative values of h while looking for the root. Thus it would be nice if we could rewrite the problem in such a way that we would never have to worry about this happening. One way is to set $h = e^{\tilde{h}}$ and solve for $\tilde{h}$ instead of h. This way h will never become negative. We can also ignore the non-binding non-negativity constraint on h. The complimentary problem now becomes: Find $\tilde{h} \in [-\infty, \ln(1)]$ such that

$$\tilde{h} > -\infty \Rightarrow (we^{\tilde{h}} + x)^{-\sigma} w - \alpha e^{\gamma \tilde{h}} \geq 0,$$

$$\tilde{h} < 0 \Rightarrow (we^{\tilde{h}} + x)^{-\sigma} w - \alpha e^{\gamma \tilde{h}} \leq 0,$$

and the minmax problem becomes

$$\min(w(we^{\tilde{h}} + x)^{-\sigma} - \alpha e^{\gamma \tilde{h}}, -\tilde{h}) = 0.$$

References

- **Miranda, Mario J. and Paul L. Fackler.** 2002. *Applied Computational Economics and Finance.* Cambridge, MA: MIT Press.