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Finite element function approximation

In these notes we consider function approximation where the basis functions do not have full support (are zero on most of the domain).

1 Piecewise linear interpolation

Let the grid be given by $\{x_1, x_2, \ldots, x_n\}$. Then we define the following "tent" functions.

$$\psi_i(x) = \begin{cases} \dfrac{x - x_{i-1}}{x_i - x_{i-1}} & \text{if } x \in [x_{i-1}, x_i] \\ \dfrac{x_{i+1} - x}{x_{i+1} - x_i} & \text{if } x \in [x_i, x_{i+1}] \\ 0 & \text{elsewhere.} \end{cases} \tag{1}$$

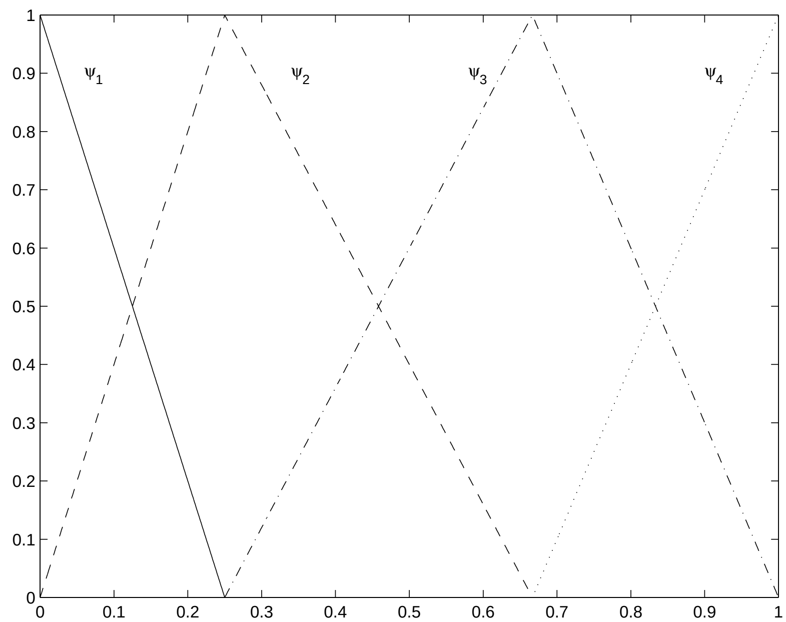

Figure 1: *Basis functions for piecewise linear interpolation.*

These functions are illustrated in Figure 1 with a grid given by $\{0, 1/4, 2/3, 1\}$. Notice that the points needn't be evenly spaced.

One may verify that if

$$\hat{f}(x) = \sum_{i=1}^{n} f(x_i)\psi_i(x)$$

then $\hat{f}(x) = f(x)$ for each $x \in \{x_1, x_2, \ldots, x_n\}$. In the notation of the handout on multi-dimensional interpolation, $\Psi = I$.

In several dimensions, life is a little bit more complicated. Then the usual approach is to let $\hat{f}$ be piecewise *multi*linear, i.e. on each element (rectangle), $\hat{f}$ is linear in each of the dimensions separately, keeping the others constant.[1]

[1]It is possible, and for some purposes preferable, to define elementwise linear functions on

We will define a piecewise bilinear approximation for the two-dimensional case. For arbitrarily many dimensions, see the handout on multi-dimensional interpolation.

Let f be a function of two variables x^1 and x^2, and let the grids be given by $\{x_1^1, x_2^1, \ldots, x_{n_1}^1\}$ and $\{x_1^2, x_2^2, \ldots, x_{n_2}^2\}$. Write ψ_i^j for the ith tent function associated with the jth grid. Then

$$\hat{f}(x^1, x^2) = \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} f(x_i^1, x_j^2)\, \psi_i^1(x^1)\, \psi_j^2(x^2).$$

Notice that this is *not* a linear function on a rectangle $[x_i^1, x_{i+1}^1] \times [x_j^2, x_{j+1}^2]$ but that it is *bi*linear on such a rectangle.

A smart way of calculating the relevant sum is based on the fact that, in any given dimension, only two of the basis functions are non-zero, and these are consecutive; call them ψ_k and ψ_{k+1}. To find k given x you use a locating algorithm that essentially proceeds by bisection.

2 Cubic splines

Piecewise linear interpolation is nice, but it yields an interpolating function that is not differentiable at the grid points. An interpolation technique that does yield differentiable approximating functions, and has other nice properties as well, is the cubic spline.

Like linear interpolation, cubic spline interpolation begins with a grid of x-values $\{x_1, x_2, \ldots, x_n\}$ and the value of the approximating function $\widehat{f}$ at those

multi-dimensional domains. But we won't do that here.

points: $\{y_1, y_2, \ldots, y_n\}$.

A cubic spline specifies $\widehat{f}$ to be a separate cubic function on each segment $[x_i, x_{i+1})$. Each cubic function has four parameters. This means that there are $4(n-1)$ parameters to determine. To do this, we impose the following conditions:

1. $\widehat{f}(x_i) = y_i; \quad i = 1, 2, \ldots, n$ (n conditions)
2. Continuity of $\widehat{f}$ at $x_2, x_3, \ldots, x_{n-1}$ ($n-2$ conditions)
3. Continuity of $\widehat{f}'$ at $x_2, x_3, \ldots, x_{n-1}$ ($n-2$ conditions)
4. Continuity of $\widehat{f}''$ at $x_2, x_3, \ldots, x_{n-1}$ ($n-2$ conditions)
5. $\widehat{f}''(x_1) = \widehat{f}''(x_n) = 0$ (2 conditions).

The final two conditions can of course easily be modified to something else depending on what is suitable in the particular case at hand.

In principle, one could impose all of these conditions at once, and the result would be a large linear system. However, this problem is best approached in two steps. First, we impose $\widehat{f}(x_i) = y_i$, continuity of $\widehat{f}$ and continuity of $\widehat{f}''$ at the gridpoints. It turns out that these requirements force us to write the cubic on each interval $[x_i, x_{i+1})$ as

$$\widehat{f}(x) = \frac{x_{i+1} - x}{x_{i+1} - x_i} \cdot y_i + \frac{x - x_i}{x_{i+1} - x_i} \cdot y_{i+1} +$$

$$+\frac{1}{6}(x_{i+1} - x_i)^2 \left\{ a_i \left[\left(\frac{x_{i+1} - x}{x_{i+1} - x_i} \right)^3 - \frac{x_{i+1} - x}{x_{i+1} - x_i} \right] + a_{i+1} \left[\left(\frac{x - x_i}{x_{i+1} - x_i} \right)^3 - \frac{x - x_i}{x_{i+1} - x_i} \right] \right\}$$

where a_i; $i = 1, 2, \ldots, n$ remain to be determined. With this specification, it turns out that $\widehat{f}''(x_i) = a_i$. We now proceed to determine these second derivatives. We do it by imposing continuity of $\widehat{f}'$ and $\widehat{f}''(x_1) = \widehat{f}''(x_n) = 0$. Continuity of $\widehat{f}'$ at the gridpoints boils down to

$$\frac{y_{i+1} - y_i}{x_{i+1} - x_i} + \frac{1}{6}(x_{i+1} - x_i)a_i + \frac{1}{3}(x_{i+1} - x_i)a_{i+1} =$$

$$= \frac{y_{i+2} - y_{i+1}}{x_{i+2} - x_{i+1}} - \frac{1}{3}(x_{i+2} - x_{i+1})a_{i+1} - \frac{1}{6}(x_{i+2} - x_{i+1})a_{i+2}$$

for $i = 1, 2, \ldots, n-2$. These equations, together with $a_1 = a_n = 0$, together constitute a tridiagonal linear system of equations. Specifically, we have $Ma = b$ where

$$b = \begin{bmatrix} 0 \\ \frac{q_2}{h_2} - \frac{q_1}{h_1} \\ \frac{q_3}{h_3} - \frac{q_2}{h_2} \\ \vdots \\ \frac{q_{n-1}}{h_{n-1}} - \frac{q_{n-2}}{h_{n-2}} \\ 0 \end{bmatrix}$$

and

$$M = \frac{1}{6} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ h_1 & 2(h_1 + h_2) & h_2 & 0 & 0 & \cdots & 0 \\ 0 & h_2 & 2(h_2 + h_3) & h_3 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & h_{n-2} & 2(h_{n-2} + h_{n-1}) & h_{n-1} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

where $q_i = y_{i+1} - y_i$ and $h_i = x_{i+1} - x_i$.

3 Shape-preserving splines

Sometimes it is important that the interpolating function preserves the "shape" of the true function. More precisely, it should be monotone wherever the true function is monotone and convex (concave) wherever the true function is convex (concave). See Schumaker (1983).

References

Schumaker, L. L. (1983, August). On shape-preserving quadratic spline interpolation. *SIAM Journal on Numerical Analysis 20*(4), 854–864.