LECTURE NOTES
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KAREN A. KOPECKY

1 Computing the Invariant Distribution

Suppose an individual's state consists of his current assets holdings, $a \in [\underline{a}, \bar{a}]$ and his current productivity level, $\lambda \in \Lambda = \{\lambda_j, j = 1, \ldots, n\}$. Hence a member of the state is a pair $(a, \lambda) \in S = [\underline{a}, \bar{a}] \times \Lambda$. In addition, suppose that next period's asset holdings are given by $a' = a'(a, \lambda)$ while next period's productivity level is governed by a first-order Markov chain with probability transition matrix P. Denote the element of P that gives the probability of obtaining λ' next period given λ today as $\pi(\lambda'|\lambda)$ or $p_{ij} = \text{Prob}(\lambda' = \lambda_j | \lambda = \lambda_i)$. Assuming it exists and is unique, how do we compute the invariant distribution of agents over the state space that is maintained in the stationary equilibrium of this economy? The invariant distribution is the distribution function, $F(a, \lambda)$ that satisfies

$$F(a', \lambda') = \sum_{\lambda \in \Lambda} \pi(\lambda'|\lambda) F(a'^{-1}(a', \lambda), \lambda) \quad (1)$$

for all $(a'\lambda') \in S$. To compute the stationary equilibrium, we can either compute the distribution function, $F(a, \lambda)$, or the density function, $f(a, \lambda)$. Four methods are described in the following subsections. Note that to implement each of the methods, we need to have an approximation of the decision rule for savings, $a'(a, \lambda)$.

1.1 Method 1: Piecewise-linear approximation of invariant distribution function

This method involves approximating F by a weighted sum of piecewise-linear functions and iterating on equation (1). Note that it is important that an approximation of F be shape-preserving since it would not make sense if F decreases over some range of a. This makes piecewise linear basis functions a better choice than say Chebyshev polynomials or cubic splines. Other approximations that preserve shape, such as shape-preserving splines could potentially also be appropriate.

First we need a grid of interpolation nodes in the interval $[\underline{a}, \overline{a}]$. Let's denote this grid as $\mathcal{A} = \{a_1 = \underline{a}, a_2, a_3, ..., a_m = \overline{a}\}$. Note that a finer grid than the grid used to compute the optimal decision rule for savings is necessary. Also the a_i's, $i = 1, \ldots, m$ do not need to be evenly-spaced along the grid.

Second we need to decide on an initial distribution over the state, $F^{(0)}$. One choice is to set

$$F_{i,j}^{(0)} = \frac{a_i - a_1}{a_m - a_1} \pi_j^*,$$

where π_j^* is the jth element of the stationary distribution for λ. Then

$$F^{(0)}(a, \lambda_j) = F_{i,j}^{(0)} + \frac{F_{i+1,j}^{(0)} - F_{i,j}^{(0)}}{a_{i+1} - a_i}(a - a_i) \quad \text{for} \quad a_i \leq a \leq a_{i+1}.$$

This is equivalent to making the initial distribution of wealth uniform across the state. Notice that, in a sense, $F^{(0)}$ is a distribution function over the assets and a density function over the shocks since

$$F^{(0)}(\tilde{a}, \lambda_j) = \text{Prob}((a \leq \tilde{a}) \cap (\lambda = \lambda_j)),$$

and

$$F^{(0)}(a_m, \lambda_j) = \pi_j^*,$$

for $j = 1, \ldots, n$ while

$$\sum_{j=1}^{n} F^{(0)}(a_m, \lambda_j) = 1.$$

Another choice is to set

$$F_{i,j}^{(0)} = \begin{cases} \pi_j^* & \text{for } a_i \geq a_{ss}, \\ 0 & \text{otherwise}, \end{cases}$$

i.e., all agents hold the representative agent economy deterministic steady state level of wealth a_{ss}.

Now compute the updated approximation to F, $F^{(1)}$ as follows

$$F_{i,j}^{(1)} = \sum_{k=1}^{n} p_{k,j} F^{(0)}(a'^{-1}(a_i, \lambda_k), \lambda_k), \tag{2}$$

then

$$F^{(1)}(a, \lambda_j) = F_{i,j}^{(1)} + \frac{F_{i+1,j}^{(1)} - F_{i,j}^{(1)}}{a_{i+1} - a_i}(a - a_i) \quad \text{for} \quad a_i \leq a \leq a_{i+1}.$$

Finally compare $F^{(0)}$ with $F^{(1)}$. If their difference is small enough then stop, otherwise set $F^{(0)} = F^{(1)}$ and repeat.

Note that computing $F^{(1)}$ requires the inverse of the decision rule for assets, $a'^{-1}(a', \lambda)$. Since, in general, for a given point in the discrete state space (a_i, λ_j) the level of assets satisfying $a = a'^{-1}(a_i, \lambda_j)$ will not be on the asset grid, we need to decide how to find it. One approach is to approximate a'^{-1} as a piecewise linear function, i.e.,

$$a'^{-1}(a_i, \lambda) = a_l + \frac{a_{l+1} - a_l}{a_{i+1} - a_i}(a - a_i),$$

where l is such that $a'(a_l, \lambda) \leq a_i \leq a'(a_{l+1}, \lambda)$.

Also note that, in general, a' may not be invertible. This occurs for example when the agent is borrowing constrained. The smallest value in the discrete grid for a effectively acts as a borrowing constraint. As such, there could be a range of possible values for a on the grid for which the optimal decision rule is to borrow such that $a' = \underline{a}$. We will therefore define the value of the inverse decision rule when $a' = \underline{a}$ to be the maximum a such that $a'(a, \lambda) = \underline{a}$.

In each iteration, two conditions are imposed. First, if $a'^{-1}(a_i, \lambda_k) < \underline{a}$ then $F(a_i, \lambda_k) = 0$. Second, if $a'^{-1}(a_i, \lambda_k) \geq \overline{a}$ then $F(a_j, \lambda_k) = \pi_k^*$. The first condition ensures that the number of agents with wealth below $\underline{a}$ is 0 and the second condition states that the number of agents with current-period wealth equal to or below $\overline{a}$ and current productivity level λ_k is equal to the number individuals with current productivity level λ_k.

1.2 Method 2: Discretization of the invariant density function

A simpler approach involves finding an approximation to the invariant density function, $f(a, \lambda)$ as opposed to the invariant cumulative distribution. We will approximate the density by a probability distribution function defined over a discretized version of the state space. Hence we will need to discretize the asset space. Denote the grid by $\mathcal{A} = \{a_1 = \underline{a}, a_2, \ldots, a_m = \overline{a}\}$. Once again the grid should be finer than the one used to compute the optimal savings rule. But suppose, for the moment, that for each point in the asset grid, a_l, $l = 1, \ldots, m$, given $\lambda \in \Lambda$ we know $a'(a_l, \lambda)$ and it is some point on the grid, say i. Our approximation to the invariant density will just be an $m \times n$ matrix of values of the density at each point in the discrete state space. To find it proceed as follows.

First initialize $f^{(0)}_{i,j} = 0$ for all $i = 1, \ldots, m$ and all $j = 1, \ldots, n$ and $f^{(0)}_{1,1} = 1$. Then compute the updated value of the density

$$f^{(1)}_{i,j} = \sum_{k=1}^{n} \sum_{l \in \mathcal{L}_i} p_{k,j} f^{(0)}_{l,k} \tag{3}$$

where $\mathcal{L}_i = \{l \in \{1, \ldots, m\} | a_i = a'(a_l, \lambda_k)\}$. In other words, do the following:

1. Set $f^{(1)}_{i,j} = 0$ for all $i = 1, \ldots, m$ and $j = 1, \ldots, n$.
2. For each $k = 1, \ldots, n$, each $l = 1, \ldots, m$ and each $j = 1, \ldots, n$ update $f^{(1)}$ as follows

$$f^{(1)}_{i,j} = f^{(1)}_{i,j} + p_{k,j} f^{(0)}_{l,k},$$

   where i is such that $a'(a_l, \lambda_k) = a_i$.

Now compare $f^{(0)}$ and $f^{(1)}$. If they are close enough then stop, otherwise set $f^{(0)} = f^{(1)}$ and repeat.

Now let's consider the case where the current asset grid $\mathcal{A}$ and the one used to compute the optimal policy rule do not coincide. Now for each pair $(a, \lambda) \in \mathcal{A} \times \Lambda$ we will need to obtain $a'(a, \lambda)$. If we only know a' for a discrete set of points, one way to do this is by linear interpolation. The algorithm is as follows.

First initialize $f^{(0)}_{i,j} = 0$ for all $i = 1, \ldots, m$ and all $j = 1, \ldots, n$ and $f^{(0)}_{1,1} = 1$. Then compute the updated value of the density

$$f^{(1)}_{i,j} = \sum_{k=1}^{n} p_{k,j} \left[\sum_{l \in \mathcal{L}_{i,1}} \frac{a_{i+1} - a'(a_l, \lambda_k)}{a_{i+1} - a_i} f^{(0)}_{l,k} + \sum_{l \in \mathcal{L}_{i,2}} \frac{a'(a_l, \lambda_k) - a_{i-1}}{a_i - a_{i-1}} f^{(0)}_{l,k} \right] \tag{4}$$

where $\mathcal{L}_{i,1} = \{l \in \{1, \ldots, m\} | a_i \leq a'(a_l, \lambda_k) \leq a_{i+1}\}$ and $\mathcal{L}_{i,2} = \{l \in \{1, \ldots, m\} | a_{i-1} \leq a'(a_l, \lambda_k) \leq a_i\}$. In other words, do the following:

1. Set $f^{(1)}_{i,j} = 0$ for all $i = 1, \ldots, m$ and $j = 1, \ldots, n$.
2. For each $k = 1, \ldots, n$, each $l = 1, \ldots, m$ and each $j = 1, \ldots, n$ update $f^{(1)}$ as follows

$$f^{(1)}_{i,j} = f^{(1)}_{i,j} + p_{k,j} \frac{a_{i+1} - a'(a_l, \lambda_k)}{a_{i+1} - a_i} f^{(0)}_{l,k}$$

$$f^{(1)}_{i+1,j} = f^{(1)}_{i+1,j} + p_{k,j} \frac{a'(a_l, \lambda_k) - a_i}{a_{i+1} - a_i} f^{(0)}_{l,k}$$

   where i is such that $a_i \leq a'(a_l, \lambda_k) \leq a_{i+1}$.

Now compare $f^{(0)}$ and $f^{(1)}$. If they are close enough then stop, otherwise set $f^{(0)} = f^{(1)}$ and repeat.

Notice that we can think of this way of handling the discrete approximation to the density function as forcing the agents in the economy to play the lottery where the probability of going to a_i given that your optimal policy is to go to $a'(a, \lambda) \in [a_i, a_{i+1}]$ is given by $\frac{a_{i+1} - a'(a,\lambda)}{a_{i+1} - a_i}$ and with probability 1 minus that you will go to a_{i+1}.

1.3 Method 3: Monte Carlo simulation

Another way to compute the invariant distribution is by means of Monte Carlo simulation. To do this one must generate a large sample of households and track them over time. Monte Carlo simulation is memory and time consuming. Hence it is not recommended for low-dimensional problems. It becomes a valuable method however when the dimension of the problem is large since it is not subject to the curse of dimensionality that plagues the other methods.

The method works as follows. First choose a sample size N. A value of N in the tens of thousands is appropriate. Next initialize the sample by assigning an initial wealth level $a^{(0),i}$ and an initial productivity level $\lambda^{(0),i}$ to each $i = 1, \ldots, N$. For example, give each agent the wealth level of the representative agent in the deterministic steady state of a corresponding representative-agent economy and draw each agent's initial productivity level from the stationary distribution for λ. Compute a set of statistics for the sample, for example, the first couple moments of a and λ. Call them $M^{(0)}$. Then update the sample. Updating asset holdings is easy since we already have the optimal savings rule, just set $a^{(1),i} = a'(a^{(0),i}, \lambda^{(0),i})$. To update λ, however, we will need a random number generator. Use a generator that gives us uniformly distributed random numbers in the interval $[0, 1]$. Suppose that at some iteration k and for individual i with last period productivity level $\lambda^{(k-1),i} = \lambda_l$ the random number is ν. Then $\lambda^{(k),i} = \lambda_{\hat{l}}$ where $\hat{l}$ is the smallest integer such that

$$\nu \leq \sum_{j=1}^{\hat{l}} p_{l,j}.$$

Compute $M^{(1)}$, the statistics of the updated sample. If $M^{(0)}$ and $M^{(1)}$ are close enough stop. Otherwise generate $a^{(2),i}$ and $\lambda^{(2),i}$ for $i = 1, \ldots, N$ and $M^{(2)}$ and continue until the M's converge.

References

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