Time-homogeneous Markov chains with finite state space in discrete time

Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space, let \(\mathcal{X} = \{1, 2, \ldots, N\}\) be a finite set and let \(X_t(\omega)\) be a stochastic process in discrete time such that \(X_t(\omega) \in \mathcal{X}\) for all \(t = 0, 1, 2 \ldots\) and all \(\omega \in \Omega\). Suppose there is an \(N \times N\) matrix \(P\) (called the probability transition matrix or PTM of the process) such that for all \(i, j \in \mathcal{X}\) we have

\[
\mathbb{P}[X_{t+1} = i | X_t = j] = P_{i,j}.
\]

Then we call the process \(X_t\) a time-homogeneous Markov chains with finite state space in discrete time.

Evidently the matrix \(P\) has non-negative entries and its columns sum to 1. We call such a matrix a stochastic matrix.

**Definition 1** Let \(P\) be a stochastic matrix. It is said to be irreducible if for every pair \(i, j \in \mathcal{X}\) there is an \(n \geq 1\) such that \((P^n)_{i,j} > 0\).

**Definition 2** Let \(P\) be an \(N \times N\) stochastic matrix and let \(1 \leq i \leq N\). The set of return times for state \(i\) is defined via

\[
R(i) = \{n \in \mathbb{N} : n > 0 \& (P^n)_{ii} > 0\}.
\]
Definition 3 Let $P$ be an $N \times N$ stochastic matrix, let $1 \leq i \leq N$ and let $R(i)$ be the set of return times for state $i$. Then the period of state $i$ is defined via

$$p(i) = \gcd R(i)$$

where $\gcd$ stands for “greatest common divisor”.

Definition 4 An $N \times N$ stochastic matrix $P$ is called aperiodic if $p(i) = 1$ for each $i = 1, 2, \ldots, N$.

Proposition 1 Each of the following two statements is equivalent to the statement that $P$ is irreducible and aperiodic.

1. $P^n$ is irreducible for each $n \geq 1$.

2. There is an $n$ such that all entries of $P^n$ are strictly positive.

Theorem 2 (Perron-Frobenius) Let $P$ be an irreducible stochastic matrix. Then 1 is an eigenvalue for $P$ and the associated eigenspace $S$ is one-dimensional. There exists a $\pi \in S = \{x \in \mathbb{R}^N : Px = x\}$ such that all entries of $\pi$ are strictly positive. There is a unique $\pi \in S$ such that $\sum_{i=1}^{N} \pi_i = 1$ and its entries are strictly positive. We call this $\pi$ the stationary probability measure associated with $P$. If $P$ is also aperiodic, then all the other eigenvalues $\lambda$ of $P$ satisfy $|\lambda| < 1$.

Proof. Omitted. ■

Corollary 3 Let $P$ be irreducible and aperiodic and let $\pi$ be its stationary probability measure. Then for every $\mu \in \mathbb{R}^N$ satisfying $\sum_{i=1}^{N} \mu_i = 1$ we have

$$\lim_{n \to \infty} P^n \mu = \pi.$$
Proof. Exercise. ■

**Theorem 4 (Ergodic theorem)** Let $X_t(\omega)$ be a time-homogeneous Markov chain with finite state space $\mathcal{X}$ in discrete time. Suppose its PTM $P$ is irreducible and let $\pi$ be its unique stationary probability measure. Let $f : \mathcal{X} \to \mathbb{R}$ be a function. Then, with probability 1,

$$
\lim_{T \to \infty} \frac{1}{T} \sum_{t=0}^{T-1} f(X_t) = \sum_{j=1}^{N} f(j) \pi_j.
$$

Proof. Omitted. ■