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Time-homogeneous Markov chains with finite state space in discrete time

Let $(\Omega, \mathcal{F}, \P)$ be a probability space, let $\mathcal{X} = \{1, 2, \ldots, N\}$ be a finite set and let $X_t(\omega)$ be a stochastic process in discrete time such that $X_t(\omega) \in \mathcal{X}$ for all $t = 0, 1, 2 \ldots$ and all $\omega \in \Omega$. Suppose there is an $N \times N$ matrix P (called the *probability transition matrix* or PTM of the process) such that for all $i, j \in \mathcal{X}$ we have

$$\P[X_{t+1} = i | X_t = j] = P_{i,j}.$$

Then we call the process X_t a *time-homogeneous Markov chains with finite state space in discrete time*.

Evidently the matrix P has non-negative entries and its columns sum to 1. We call such a matrix a *stochastic matrix*.

Definition 1 *Let P be a stochastic matrix. It is said to be* irreducible *if for every pair $i, j \in \mathcal{X}$ there is an $n \geq 1$ such that $(P^n)_{i,j} > 0$.*

Definition 2 *Let P be an $N \times N$ stochastic matrix and let $1 \leq i \leq N$. The* set of return times for state i *is defined via*

$$R(i) = \{n \in \mathcal{N} : n > 0 \ \& \ (P^n)_{ii} > 0\}.$$

Definition 3 *Let P be an $N \times N$ stochastic matrix, let $1 \leq i \leq N$ and let $R(i)$ be the set of return times for state i. Then the* period *of state i is defined via*

$$p(i) = \gcd R(i)$$

where gcd *stands for "greatest common divisor".*

Definition 4 *An $N \times N$ stochastic matrix P is called* aperiodic *if $p(i) = 1$ for each $i = 1, 2, \ldots, N$.*

Proposition 1 *Each of the following two statements is equivalent to the statement that P is irreducible and aperiodic.*

1. *P^n is irreducible for each $n \geq 1$.*
2. *There is an n such that all entries of P^n are strictly positive.*

Theorem 2 (Perron-Frobenius) *Let P be an irreducible stochastic matrix. Then* 1 *is an eigenvalue for P and the associated eigenspace S is one-dimensional. There exists a $\pi \in S = \{x \in \mathcal{R}^N : Px = x\}$ such that all entries of π are strictly positive. There is a unique $\pi \in S$ such that $\sum_{i=1}^N \pi_i = 1$ and its entries are strictly positive. We call this π the* stationary probability measure *associated with P. If P is also aperiodic, then all the other eigenvalues λ of P satisfy $|\lambda| < 1$.*

Proof. Omitted. ■

Corollary 3 *Let P be irreducible and aperiodic and let π be its stationary probability measure. Then for every $\mu \in \mathcal{R}^N$ satisfying $\sum_{i=1}^N \mu_i = 1$ we have*

$$\lim_{n \to \infty} P^n \mu = \pi.$$

Proof. Exercise. ■

Theorem 4 (Ergodic theorem) *Let $X_t(\omega)$ be a time-homogeneous Markov chain with finite state space $\mathcal{X}$ in discrete time. Suppose its PTM P is irreducible and let π be its unique stationary probability measure. Let $f : \mathcal{X} \to \mathcal{R}$ be a function. Then, with probability 1,*

$$\lim_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} f(X_t) = \sum_{j=1}^{N} f(j)\pi_j.$$

Proof. Omitted. ■