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Introduction to stationary probability measures

1 Motivation

Sometimes in economic models, variables move around at random over time. We might then be interested in whether the variables converge not to a limiting value but a limiting distribution. Leaving aside the precise mathematical formulation of this for the time being, we note here that the notion of a limiting distribution has two interpretations, a macro and a micro. The macro interpretation is that the probabilities of the variables at t being in various sets A converge as t tends to infinity. The micro interpretation is that the fractions of individuals with their characteristics in various sets A converge as t tends to infinity.

2 A concrete (macro) example

Consider a simple version of the stochastic growth model, where a planner maximizes

$$\mathsf{E}\left[\sum_{t=0}^{\infty}\beta^t \ln c_t\right]$$

subject to

$$\begin{cases} c_t + k_{t+1} & \leq & z_t k_t^\alpha \\ k_{t+1} & \geq & 0 \\ \ln z_{t+1} & = & \rho \ln z_t + \varepsilon_{t+1} \end{cases}$$

and $k_0 > 0$ given where ε_{t+1} is i.i.d. normal with mean 0 and variance σ_ε^2. Also, it is a constraint that k_{t+1} and c_t cannot be contingent on ε_{t+1}, ε_{t+2} etc. We assume that $0 < \alpha < 1$, that $-1 < \rho < 1$ and that $0 \leq \beta < 1$.

One may verify that the optimal solution can be represented as

$$k_{t+1} = \alpha\beta z_t k_t^\alpha$$

or, equivalently,

$$\ln k_{t+1} = \ln(\alpha\beta) + \ln z_t + \alpha \ln k_t.$$

For some purposes we may want to write this as

$$\ln k_{t+1} = g(\ln k_t, \ln z_t).$$

Now, to begin with, we would like to know something about the distribution of $(\ln k_{t+1}, \ln z_{t+1})$ given $(\ln k_t, \ln z_t)$ given. For that purpose, define

$$S = \mathcal{R} \times \mathcal{R}$$

with typical element $s \in S$ with the interpretation $s = (\ln k, \ln z)$. Define $\mathcal{B}$ as the Borel σ-algebra on S. Now we want to know the distribution of s' given s. To achieve that, we introduce the concept of a Probability Transition Function (PTF), denoted by $Q(s, A)$.

We will be (even) more formal about this later, but the interpretation is that $Q(s, A)$ is the probability that the next period's state $s' \in A$ given that the current state is s.

What does the PTF look like in this particular case? Well, $\ln k'$ is known to equal $g(s)$. So let $A = A_1 \times A_2$. Not all Borel sets A can be written in this way, but our strategy will be to define the PTF on those sets that can, and then extend uniquely by additivity.

$$Q(s, A) = \begin{cases} 0 & \text{if} \quad g(s) \notin A_1 \\ \displaystyle\int_{A_2} \varphi_{\rho \ln z, \sigma_\varepsilon}(x) dx & \text{if} \quad g(s) \in A_1 \end{cases}$$

where $\varphi_{\mu,\sigma}$ is the density function for the normal distribution with mean μ and standard deviation σ.

Denote the probability distribution of s_t by $\mu_t(A)$. Suppose we start with an initial probability distribution on s. Call it $\mu_0(A)$. (It could be degenerate so that s_0 is known with certainty.) We then have the following difference equation determining the sequence $\{\mu_t\}$.

$$\mu_{t+1} = \int_A Q(s, A) d\mu_t(s)$$

and we are interested in whether and to what the sequence $\{\mu_t\}$ might converge, leaving aside the issue of what this might mean exactly.

What we will notice in this special case, though, is that we don't have to integrate over distributions; in each period, the distribution is fully characterized by the mean and the variance, since it is normal (provided the initial period distribution is normal). This follows from the following observation, which follows from the fact that linear combinations of normal random variables are normal.

If $(\ln k, \ln z)$ is normally distributed with mean (μ_k, μ_z) and variance matrix

$$\begin{bmatrix} \sigma_k^2 & \sigma_{kz} \\ \sigma_{kz} & \sigma_z^2 \end{bmatrix}$$

then $(\ln k', \ln z')$ is normally distributed with mean

$$\begin{bmatrix} \ln(\alpha\beta) + \mu_z + \alpha\mu_k \\ \rho\mu_z \end{bmatrix}$$

and variance

$$\begin{bmatrix} \sigma_z^2 + \alpha^2\sigma_k^2 + 2\alpha\sigma_{kz} & \alpha\rho\sigma_{kz} + \rho\sigma_z^2 \\ \alpha\rho\sigma_{kz} + \rho\sigma_z^2 & \rho^2\sigma_z^2 + \sigma_\varepsilon^2 \end{bmatrix}$$

Setting the current and next period means equal, we get a limiting mean of

$$\begin{bmatrix} \frac{\ln(\alpha\beta)}{1-\alpha} \\ 0. \end{bmatrix}$$

Setting current and next period variances equal, we get a limiting variance of

$$\frac{\sigma_\varepsilon^2}{1-\rho^2} \cdot \begin{bmatrix} \frac{1+\alpha\rho}{(1-\alpha^2)(1-\alpha\rho)} & \frac{\rho}{1-\alpha\rho} \\ \frac{\rho}{1-\alpha\rho} & 1 \end{bmatrix}.$$