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Lecture 5: General equilibrium under exogenously imposed borrowing constraints

1 Introduction

The empirical relevance of incomplete capital/insurance markets is documented in Attanasio and Davis (1996) and Attanasio (1999). Another case for assuming imperfect insurance markets is that it leads to a theory with definite implications for the stationary distribution of wealth.

For decision theory, see Lecture 3.

2 Invariant measures and all that

Suppose we are in a situation where something (say income or wealth) gets jumbled in a random but systematic way across individuals in each period. Random, because no one knows exactly who is going to get rich quick and who is going to end up on skid row. Systematic, because the same fraction of agents switch from a given level to another given level in each period.

Under what circumstances can we be sure that the economy will converge to a unique distribution of income or wealth? That is the topic of this section. It turns out that the following three conditions will be sufficient:

1. The Feller property, for existence.
2. The mixing property, for uniqueness.
3. Monotonicity, for convergence.

We begin by describing formally what it means to jumble in a random but systematic way.

Let (S, d) be a metric space. Let τ be the topology generated by d and let $\mathcal{B}$ be the Borel σ–algebra generated by τ. Let $Q(s, A)$ be a mapping from $S \times \mathcal{B}$ into $[0, 1]$ such that

1. For each fixed $s \in S$, $Q(s, \cdot)$ is a probability measure and
2. For each fixed A, $Q(\cdot, A)$ is a $\mathcal{B}$–measurable function.

We call such a Q a **probability transition function**. In what follows, we will essentially be talking about the following difference equation

$$\mu_{t+1}(A) = \int_S Q(s, A) d\mu_t(s) \tag{1}$$

where $\{\mu_t\}$ is a sequence of probability measures on $(S, \mathcal{B})$. We are interested in the conditions under which a sequence $\{\mu_t\}$ satisfying (1) converges to a unique limiting probability measure, independently of the initial probability measure μ_0.

Having defined the transition function $Q(s, A)$, we can define other objects in terms of it. For example, suppose s is transformed into s' according to Q and you want to know $\mathsf{E}[f(s')|s]$. Or you want to know the average of $f(s)$ when s is distributed according to μ. Or the average of $f(s')$ when s is distributed according to μ.

Define the operators T and T^* in terms of Q via the following. Let $\mu : \mathcal{B} \to [0, 1]$ be a probability measure and let $f : S \to \mathbb{R}$ be a bounded or non–negative measurable function.

$$(Tf)(s) = \int\limits_S f(y)Q(s, dy)$$

$$(T^*\mu)(A) = \int\limits_S Q(s, A)d\mu(s).$$

Moreover, we sometimes write

$$(f, \mu) = \int\limits_S f d\mu.$$

Definition. The function Q is said to have the **Feller property** if

$$(Tf)(s) = \int\limits_S f(y)Q(s, dy)$$

is bounded and continuous whenever f is bounded and continuous.

Proposition. The following properties are equivalent to the Feller property.

1. $s_n \to s$ implies $Q(s_n, \cdot)$ converges weakly to $Q(s, \cdot)$.
2. μ_n converges weakly to μ implies that $T^*\mu_n$ converges weakly to $T^*\mu$.
3. For each open set $A \in \tau$, the function $Q(\cdot, A)$ is lower semicontinuous, i.e. for each $a \in [0, 1]$, the set $\{s \in S : Q(s, A) > a\}$ is open.

We will define the notion of weak convergence of measures in a moment. But before we do that, we will define the central concept of this Lecture: an invariant measure.

Definition. A Q–invariant measure is a probability measure such that, for all $B \in \mathcal{B}$, we have

$$\mu(B) = \int_S Q(s, B) d\mu(s).$$

In shorthand notation:

$$T^* \mu = \mu.$$

We have the following existence result for invariant measures.

Proposition. Let (S, d) be a compact metric space and suppose Q has the Feller property. Then there is an invariant measure.

Proof. See Aliprantis and Border (1994). ■

To understand better what this means, let's consider two cases where there does not exist an invariant measure.

Example. Let $S = \mathbb{N} = \{1, 2, 3, \ldots\}$ and define d via

$$d(n, m) = |n - m|.$$

Notice that this metric generates the discrete topology. Let Q be given by

$$Q(n, A) = \begin{cases} 1 \text{ if } (n+1) \in A \\ 0 \text{ otherwise.} \end{cases}$$

This Q has no invariant measure. This is possible since S is not compact. However, we can make it compact by adding one point and defining a new metric.

Example. Let $S = \mathbb{Z}_+ = \{0, 1, 2, 3, \ldots\}$ and define d via

$$d(n, m) = \begin{cases} \left|\frac{1}{m} - \frac{1}{n}\right| & \text{if } n \neq 0 \text{ and } m \neq 0 \\ \frac{1}{m} & \text{if } n = 0 \text{ and } m \neq 0 \\ \frac{1}{n} & \text{if } n \neq 0 \text{ and } m = 0 \\ 0 & \text{if } n = 0 \text{ and } m = 0. \end{cases}$$

The topology generated by this metric is the one associated with the Alexandrov one–point compactification of $\mathbb{N}$ with 0 as the point at infinity. That is, the open sets are all the subsets of $\mathbb{N}$ and sets of the form

$$\{0\} \cup K^c$$

where $K \subset \mathbb{N}$ is a finite set.[1] It follows that (S, d) is a compact metric space. Notice that all the singletons are open sets except $\{0\}$, which is not open. No matter how small you choose $\varepsilon > 0$, there is a point $n \in S$ such that $d(0, n) < \varepsilon$ but $n \notin \{0\}$.

Now let Q be defined as in the previous example and it should be fairly clear that, even though S is compact, there is still no invariant measure. (For a proof of this, see Aliprantis and Border (1994).) So it must be that Q is not Feller. We will show this in two ways. First, we show that there is an open set $A \subset S$ such that $Q(\cdot, A)$ is not lower semicontinuous. Choose $A = \{1\}$. This is an open set. Meanwhile, the inverse image of $(\frac{1}{2}, 1]$ under $Q(\cdot, A)$ is the set $\{0\}$, which is not open.

On the other hand, consider the bounded and continuous function $f : S \to \mathbb{R}$ defined via

$$f(n) = \begin{cases} \frac{1}{n} & \text{if } n \neq 0 \\ 0 & \text{if } n = 0. \end{cases}$$

(The reader should verify that this really is a continuous function.) Meanwhile, applying the operator T to this function, we get

$$(Tf)(n) = \frac{1}{n+1}.$$

This is not a continuous function. To see this, consider the inverse image of the open set $(\frac{3}{4}, \frac{5}{3})$. Alternatively, consider the sequence $n_t = t$. Apparently

$$\lim_{t \to \infty} n_t = 0$$

[1] More generally, the Alexandrov one–point compactification of a topological space (X, τ) is the set $X \cup \{\infty\}$ together with the following topology. A set A is open either if $A \in \tau$ or if there is a compact set $K \subset X$ such that $K^c \in \tau$ and $A = \{\infty\} \cup K^c$. Of course, the requirement that K^c be open is redundant if (X, τ) is Hausdorff.

and

$$\lim_{t\to\infty}(Tf)(n_t) = 0.$$

Yet $(Tf)(0) = 1 \neq 0$, so Tf is not continuous at $n = 0$.

We are not just interested in existence, however. We want uniqueness and convergence, too. By convergence, we mean that, if we define the probability measure μ_0 arbitrarily and define

$$\mu_{t+1} = T^*\mu_t$$

then μ_t will converge to a probability measure μ such that $T^*\mu = \mu$. But what are we to mean by convergence? After all, we haven't defined a topology for measures, although we could (see Aliprantis and Border (1994)).

Stupid definition. The sequence of measures $\{\mu_n\}$ is said to converge to the limit measure μ if, for each $B \in \mathcal{B}(S)$

$$\lim_{n\to\infty}\mu_n(B) = \mu(B).$$

This is a stupid definition because, under this definition, there can be a sequence of random variables $\{X_n\}$ that converges in probability to the random variable X but the distribution measures of $\{X_n\}$ do not converge to the distribution measure of X.

Good definition. The sequence of measures $\{\mu_n\}$ is said to converge (weakly, vaguely) to the limit measure μ if, for each bounded and continuous function $f : S \to \mathbb{R}$, we have

$$\lim_{n\to\infty}\int_S f d\mu_n = \int_S f d\mu.$$

We now introduce the mixing property, which is there to ensure uniqueness.

Definition. Suppose $S = [a_1, b_1] \times [a_2, b_2] \times \ldots \times [a_n, b_n]$ is a rectangle in $\mathbb{R}^n$. We write $S = [a, b]$. Then Q is said to exhibit the **mixing property** if there is a $c \in S$ and an $N \geq 1$ such that

$$Q^N(a, [c, b]) > 0 \quad \text{and}$$

$$Q^N(b, [a, c]) > 0$$

where

$$Q^2(x, A) = \int_X Q(y, A)Q(x, dy).$$

As a counter–example, consider a two–state Markov chain with the transition probability matrix

$$\Gamma = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

so that, for sure, you stay in whatever state you started. In this case, any probability measure is invariant, so there is existence but not uniqueness.

Finally, we introduce a property that will ensure convergence. For this property (monotonicity) to be defined, we need to equip S with an **order**.

Definition. An **order** on a set X is a relation $\geq$ satisfying, for all $x, y, z \in X$,

- If $x \geq y$ and $y \geq z$ then $x \geq z$ (transitivity),
- if $x \geq y$ and $y \geq x$ then $x = y$ (antisymmetry), and
- $x \geq x$ (reflexivity).

Definition. An ordered pair $(X, \geq)$ where X is a set and $\geq$ is an order on X is called a **partially ordered set**, or just an **ordered set**.

Definition. A function $f : X \to \mathbb{R}$ where $(X, \geq)$ is an ordered set is said to be **increasing** if $f(y) \geq f(x)$ whenever $y \geq x$.

Definition. A transition function Q is said to be **monotone** if the associated operator T has the property that for any bounded, increasing real function f, Tf is also increasing.

This definition captures the idea that, if the current value is high, you are more likely to get a high value in the future as well. Positive autocorrelation, if you like. A counterexample is a two–state Markov chain with the transition probability matrix

$$\Gamma = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

In this case, there is a unique invariant measure ($\mu_1 = \mu_2 = 1/2$), but unless you start there you don't converge to it. You keep switching back and forth.

Theorem [12.12 in Stokey and Lucas (1989)]. Let $S \subset \mathbb{R}^n$ be a compact rectangle. Associate it with the Euclidean metric and the following order. $x \geq y$ if $x_i \geq y_i$ for all $i = 1, 2, \ldots, n$. If Q is monotone, has the Feller property and exhibits the mixing property then there is a unique probability measure μ such that

$$T^* \mu = \mu$$

and a sequence of measures defined via

$$\mu_{t+1} = T^* \mu_t$$

will converge weakly to μ no matter what the initial probability measure μ_0 is.

But what if S is not a rectangle (but nevertheless compact)? What matters (see Hopenhayn and Prescott (1992)) is the following

1. S is associated with an order $\geq$.
2. Every chain $A \subset S$ has a least upper bound in S.
3. S has a least element and a greatest element.

The mixing condition can then be stated in terms of the least and greatest elements as follows. Let a be the least element and b be the greatest element. Then the mixing condition is that there is a $c \in S$ and an N such that

$$Q^N(a, \{s : s \geq c\}) > 0$$

and

$$Q^N(b, \{s : s \le c\}) > 0.$$

In what I just said, there were some undefined terms (chain, least, upper bound). The pedestrian reader may want to settle for the following characterization. In metric spaces, each chain in S having a least upper bound is equivalent to the relation $\ge$ being a closed subset of $S \times S$ (recall that a relation on S is a subset of $S \times S$). But for completeness, here are the general definitions.

Definition. Let $(X, \ge)$ be an ordered set. Then $x \in X$ is said to be a **greatest** element if $x \ge y$ for all $y \in X$.

Definition. Let $(X, \ge)$ be an ordered set. Then $x \in X$ is said to be a **least** element if $y \ge x$ for all $y \in X$.

Remark. On a partially ordered set, a **maximal** element is **not** the same as a greatest (maximum) element, nor is a **minimal** element the same as a least (minimum) element.

Definition. Let $(X, \ge)$ be an ordered set. Then $x \in X$ is said to be a **maximal** element if $y \ge x$ implies $y = x$.

Definition. Let $(X, \ge)$ be an ordered set. Then $x \in X$ is said to be a **minimal** element if $x \ge y$ implies $y = x$.

Definition. An ordered set $(X, \ge)$ is said to be **totally ordered** if for all $x, y \in X$, either $x \ge y$ or $y \ge x$.

Definition. Let $(X, \ge)$ be an ordered set and let $A \subset X$. Then A is said to be a **chain** if it is totally ordered by $\ge$.

Definition. Let $(X, \ge)$ be an ordered set and let $A \subset X$. Then $x \in X$ is said to be an **upper bound** of A if $x \ge a$ for all $a \in A$.

Definition. Let $(X, \ge)$ be an ordered set and let $A \subset X$. Then $x \in X$ is said to be a (and hence the) **least upper bound** of A if it is an upper bound and $y \geq x$ for any other upper bound $y \in X$.

Example. Let $S = \{0,1\} \times [0,1]$ and equip it with the following order. $x \geq y$ if $x_1 \geq y_1$ and $x_2 \geq y_2$. This set has the property that every chain $A \subset S$ has a least upper bound in S. It also has a least element: $(0,0)$ and a greatest element: $(1,1)$. However, S is not totally ordered. With $x = (0,1)$ and $y = (1,0)$ we neither have $x \leq y$ nor $y \leq x$.

Finally, there is one more fact about probability transition measures that we need to know, namely Theorem 8.3 in Stokey and Lucas (1989).

Theorem. Let T and T^* be defined as above and let f be a non–negative measurable function on $(S, \mathcal{B})$ into $\mathbb{R}$. Then

$$\int_S (Tf) d\mu = \int_S f d(T^*\mu)$$

for any probability measure μ on $\mathcal{B}$.

Proof. By the Monotone Convergence Theorem, it suffices to show it for a simple function. We show it here for an indicator function. For the rest, see Stokey and Lucas (1989). Let $A \in \mathcal{B}$ and let $f = I_A$. Then

$$(Tf)(s) = Q(s, A)$$

and the left hand side becomes

$$\int_S Q(s, A) d\mu.$$

Meanwhile, the right hand side is

$$\int_S I_A(s) d(T^*\mu) = (T^*\mu)(A) = \int_S Q(s, A) d\mu.$$

■

3 General equilibrium and the risk free rate

Result (Huggett): When the borrowing constraint is one year's income, the risk free rate is more than one percent below the rate in the corresponding representative agent economy.

Intuition: A low risk–free rate is needed to persuade agents not to accumulate large assets so that the credit market can clear. The net supply of bonds is zero, so aggregate net demand must be zero in equilibrium. There is no storage technology or capital.

Potential importance: maybe this could go some ways towards solving the equity premium puzzle.

We now describe the equilibrium in Huggett (1993). A lower bound for asset holdings exists by assumption. An upper bound exists because $q > \beta$. Denote earnings by e_t and let them follow a two–state Markov process. There are no aggregate shocks, so $q_t = q$ for all t.

Thus the state space for an individual can be written as

$$S = [\underline{a}, \overline{a}] \times \{e_\ell, e_h\}.$$

A typical element of S is $s = (a, e)$. Let β_S denote the Borel σ–algebra on S. Let $\psi : \beta_S \to [0, 1]$ be a probability measure describing the population distribution across states.

We will be concerned steady states

$$\psi' = \psi.$$

Let $Q(s, B)$ be the probability transition measure for individuals. The probability of $s' \in B$ given s can be constructed from the decision rule $a'(s)$ and the transition probabilities $\pi(e'|e)$ as follows. This is done in a very roundabout way in Huggett's paper. It can be simplified and that's what we'll do here. We begin by defining the probability transition measure on pairs (s, B) where B can be written as $B = \{e'\} \times A$ where $A \subset [\underline{a}, \overline{a}]$ is a Borel set. For these simple sets, $Q(s, B)$ is just $\pi(e'|e)$ if $a'(s) \in A$ and zero otherwise. We can then extend the measure to other types of sets by additivity.

A steady state (or stationary equilibrium) is a measure ψ such that

$$\psi(B) = \int\limits_S Q(s, B) d\psi$$

for all $B \in \beta_S$. To summarize the definition of a stationary equilibrium, it consists of a savings function $a'(s)$, a consumption function $c(s)$, and a stationary measure ψ such that

1. $a'(s)$ and $c(s)$ are optimal given q,
2. Markets clear so that
$$\int\limits_S c(s) d\psi = \int\limits_S e(s) d\psi$$
and
$$\int\limits_S a'(s) d\psi = 0$$
3. ψ is stationary, i.e.
$$\psi(B) = \int\limits_S Q(s, B) d\psi.$$

Notice that, by Theorem 8.3 in Stokey and Lucas (1989), if $\psi = T^* \psi$, then

$$\int\limits_S a(s) d\psi = \int\limits_S a'(s) d\psi$$

i.e. total assets equal total savings.

Theorem. Let q be given and suppose it satisfies $0 \leq \beta < q$. If $\pi(e_h|e_h) \geq \pi(e_h|e_\ell)$ and $\underline{a} + e_\ell - \underline{a}q > 0$, then there is a unique stationary distribution ψ under which markets may or may not clear, and $\psi_t \to \psi$ weakly for any ψ_0.

3.1 Calibration and numerical calculations

Huggett thinks of e_h and e_ℓ as earnings when employed and not employed, respectively. The process for e is then calibrated to match facts about the variability of labour earnings and the average duration of unemployment in the United States.

With this in mind, and setting the model period length to two months, Huggett sets $e_h = 1.0$, $e_\ell = 0.1$, $\pi(e_h|e_h) = 0.925$ and $\pi(e_h|e_\ell) = 0.5$. The discount factor β is set to 0.99322, corresponding to an annual discount factor of 0.96. The coefficient of relative risk aversion, σ, is set to 1.5 or 3.0, and a range of different credit limits $\underline{a}$ are considered.

The computation method consists of three steps.

1. Given q, compute $a'(s;q)$.
2. Given $a'(s;q)$, start with an arbitrary ψ_0 and iterate on

   $$\psi_{n+1}(B) = \int_S Q(s,B) d\psi_n$$

   until approximate convergence. Use the resulting measure ψ to calculate

   $$\int_S a'(s;q) d\psi.$$

3. Update q and repeat (1) and (2) until there is approximate market clearing.

Obviously, these steps have to be described in more detail. We have talked about step (1) already, so let's move on to step (2). The measure ψ will be represented by a distribution function F.

We begin by defining the function

$$F_0(a,e) = \psi_0(\{s : s_1 \leq a, s_2 = e\})$$

on the gridpoints, and let it be defined elsewhere by linear interpolation. Then iterate on

$$F_{t+1}(a', e') = \sum_{e \in \{e_h, e_\ell\}} \pi(e'|e) F_t(a'^{-1}(\cdot, e)(a'), e)$$

on gridpoints (a', e').

What motivates this formulation? And what does it mean exactly if the function a' is not invertible? (It isn't since it is neither onto nor one-to-one.) We can imagine that t really represents time and that the above equation describes the evolution of ψ_t. How many people will have earnings e' and assets below a' tomorrow? Well, some of the ones that have e_h now and some of the ones that have e_ℓ now will have e' tomorrow — that is taken care of by the probabilities $\pi(e'|e)$. But for a given group of people who have earnings e today and will have earnings e' tomorrow, how many of then will have assets below a'? Well, it will be those that have assets a such that $a'(a, e) \leq a'$.

On the part of the domain where $a'(a, e)$ is strictly increasing in a, it is easy to characterize the asset holdings of these people. Define $\tilde{a}$ as the unique number such that $a'(\tilde{a}, e) = a'$. Then the number of people with asset holdings tomorrow below a' are just the number of people with asset holdings today below $\widetilde{a}$ and

$$F_{t+1}(a', e') = \sum_{e \in \{e_h, e_\ell\}} \pi(e'|e) F_t(\widetilde{a}, e).$$

But we know that, for $e = e_\ell$ and sufficiently low a, a' will be constant and equal to $\underline{a}$. Everyone with $e = e_\ell$ and asset holdings below the kink in the savings function will have assets $\underline{a}$ tomorrow. Call the point where the kink is $\widehat{a}$. Apparently

$$F_{t+1}(\underline{a}, e_\ell) = \sum_{e \in \{e_h, e_\ell\}} \pi(e'|e) F_t(\widehat{a}, e).$$

On the other hand, for $e = e_h$, no one will have the asset holdings $\underline{a}$ tomorrow. Thus the appropriate definition is the following.

$$a'^{-1}(\cdot, e)(a') = \sup\{a \in [\underline{a}, \overline{a}] : a'(a, e) = a'\}$$

where we adopt the convention that $\sup \varnothing = -\infty$ which is appropriate since $F(-\infty, e) = 0$.

In any case, iterations are continued until the seqence of functions $F_t(a, e)$ approximately converges. The savings function $a'(a, e)$ and the converged distribution function $F(a, e)$ is then used to check market clearing. q is then adjusted upwards if the net demand for bonds is positive, and downwards if it is negative.

To compute total assets for a given distribution function, you may want to make use of the following result. If a non–negative random variable X is distributed according to F, then its expected value equals

$$\mathsf{E}[X] = \int_0^\infty x dF(x) = \int_0^T [1 - F(x)] dx.$$

(You can prove this using integration by parts.) If X has a lower bound a (perhaps a negative one), the formula becomes

$$\mathsf{E}[X] = a + \int_a^\infty [1 - F(x)] dx$$

and if it also has an upper bound b, then this reduces to

$$\mathsf{E}[X] = b - \int_a^b F(x) dx.$$

3.1.1 An alternative way to calculate the stationary equilibrium

Discretize the state space, i.e. approximate S by a finite set. Use value function iteration to solve for $a'(s)$. This defines Q and this Q can be represented as a matrix Γ. The stationary equilibrium distribution is that eigenvector of Γ associated with the unit eigenvalue whose entries sum to one. Alternatively, just take any vector μ_0 whose entries sum to 1 and compute $\lim_{t\to\infty}\Gamma^t\mu_0$. A slow way to do this is to use the recursion $\mu_{t+1} = \Gamma\mu_t$. A quicker way to do this is to set $M_0 = \Gamma$ and iterate on

$$M_{t+1} = M_t^2.$$

This is called a "doubling algorithm". Denote the limit matrix by M. Then the stationary equilibrium is $\mu = M\mu_0$.

3.2 Results

To interpret the meaning of a given borrowing constraint $\underline{a}$, note that an average year's earnings is -5.3.

See Tables 1 and 2 of the paper.

4 Borrowing constraints in the growth model

The model of the previous section was an endowment economy with no storage or capital. There was no precautionary saving **in the aggregate**. In Huggett (1990), on the other hand, the usual neoclassical production function is brought back in and the model now has implications for total saving. In this context, the borrowing constraint is set to zero so that no one can own a negative amount of capital.

It is shown that the capital stock is larger in an economy with borrowing constraints than in the corresponding complete–markets economy. The proof uses Theorem 8.3 in Stokey and Lucas (1989).

Here is an outline of the proof. The result is that, in a steady state, under standard assumptions, the (gross) marginal product of capital is lower than it would be under complete markets. Notice that Huggett defines the production function f so that $f(K)$ is total available resources, including the undepreciated part of the capital stock.

We begin by defining aggregate capital via the following

$$K(\psi) = \int_X k d\psi$$

where $x = (k, e)$ and ψ is the stationary measure.

The proof has two steps. The first is to show that

$$\beta f'(K(\psi)) \leq 1.$$

The second is to show that

$$\beta f'(K(\psi)) \neq 1.$$

To show the first part, note that the following is a necessary condition for consumer optimization.

$$u'(c(x)) \geq \beta(1+r)\mathsf{E}[u'(c(k(x), e'))|x].$$

Now integrate both sides of the equation with respect to the stationary measure ψ. We get

$$\int_X u'(c(x))d\psi \geq \beta(1+r)\int_X \mathsf{E}[u'(c(k'(x), e'))|x]d\psi.$$

Now recall Theorem 8.3 in Stokey and Lucas (1989) and note that it says that, for any non–negative measurable function $f : X \to \mathbb{R}$, we have

$$\int_X \mathsf{E}[f(x)|x]d\psi = \int_X f(x)d\psi^*$$

where

$$\mathsf{E}[f(x)|x] = \int_X f(y)Q(x, dy)$$

and

$$\psi^*(A) = \int_X Q(x, A)d\psi(x).$$

But since ψ is stationary, we have $\psi^* = \psi$ so

$$\int_X \mathsf{E}[f(x)|x]d\psi = \int_X f(x)d\psi.$$

But then

$$\int_X \mathsf{E}[u'(c(k'(x), e'))|x]d\psi = \int_X u'(c(x))d\psi$$

and consequently

$$\int_X u'(c(x)))d\psi \geq \beta(1+r) \int_X u'(c(x)))d\psi$$

from which it follows that

$$1 \geq \beta(1+r) = \beta f'(K(\psi)).$$

Showing that $\beta f'(K(\psi)) \neq 1$ is considerably more technical, and the reader is referred to the paper.

5 Wealth distribution in a life–cycle economy

The idea in Huggett (1996) is to calibrate the model to match certain features of the U.S. earnings distribution and then examine the implications for the wealth distribution.

The environment is an overlapping generations model where agents have realistic life–spans. They experience variations in earnings for both deterministic and idiosyncratically stochastic reasons. They save for retirement and for precautionary reasons.

There is just one asset in the usual sense of the word: physical capital. Since there are no aggregate shocks, the return on capital is riskless.

Also, there is a pension scheme ("social security system"). We will **not** include anticipated social security payments in wealth.

People live for a maximum of N periods and face a survival probability s_t conditional on having survived until period $t-1$. The population grows at rate n.

Each person maximizes

$$\mathsf{E}\left[\sum_{t=1}^{N}\beta^t\left(\prod_{j=1}^{t}s_j\right)u(c_t)\right]$$

where

$$u(c)=\frac{c^{1-\sigma}}{1-\sigma}.$$

Each person is endowed with labour according to the function $e(z,t)$ where z is the current realization of a Markov process and t is age.

Aggregate output is produced according to

$$Y=F(K,L)=AK^{\alpha}L^{1-\alpha}.$$

We assume that a law of large numbers applies so that L is constant over time. If everyone lives until the age N, then a formula for L is given by

$$L=\frac{n(n+1)}{(1+n)^N-1}\sum_{k=1}^{N}(1+n)^k e(\mathsf{E}[z],k).$$

Things are a bit more complicated if $s_t<1$ for some t.

The (flow) budget constraint is given by

$$c+a'\leq a[1+r(1-\tau)]+(1-\theta-\tau)e(z,t)w+T+b_t$$

where T is a lump–sum transfer and b_t is an age–dependent (but earnings–independent) social security transfer payment. Capital and labour income are taxed at rate τ. In addition, there is a social security tax θ.

An agent also faces the following constraints: $c\geq 0$, $a'\geq \underline{a}$ and, if $t=N$, $a'\geq 0$.

Agents retire at age R so that $e(z,R)=0$, $b_{R-1}=0$ and $b_R=b$ etc.

To define the equilibrium, we need som measure theory. Let each agent be described by her age, her asset position a and her labour endowment shock z. Consider the following measure space.

$$(X, \mathcal{B}, \psi_t)$$

where $X=[\underline{a},\infty)\times Z$, $\mathcal{B}$ is the Borel σ–algebra of subsets of X and ψ_t is a probability measure such that $\psi_t(B)$ is the fraction of age t agents whose (a,z) lies in the set $B\in\mathcal{B}$.

Meanwhile, there is a measure μ_t which describes the fraction of the population that is of age t. Then $\mu_t\psi_t(B)$ is the fraction of the total population who are age t and whose (a,z) lie in B.

Now define $Q(x,t,B)$ as the probability that an agent of age t transits from x to B, and we know from above how to construct such a transition measure.

We are now in a position to define a stationary equilibrium. Let $c(x,t)$ be a consumption function and let $a'(x,t)$ be a savings function and let r, w, K, L, G, τ, θ, b be numbers and let $\psi_1,\psi_2,\ldots\psi_N$ be probability measures such that

1. $c(x,t)$ and $a'(x,t)$ are optimal
2. $w=F_2(K,L)$ and $r=F_1(K,L)-\delta$
3. Markets clear, i.e.

$$\sum_{t=1}^{N}\mu_t\int_X[c(x,t)+a'(x,t)]d\psi_t+G=F(K,L)+(1-\delta)K,$$

$$\sum_{t=1}^{N}\mu_t\int_X a'(x,t)d\psi_t=(1+n)K$$

and

$$\sum_{t=1}^{N}\mu_t\int_X e(z,t)d\psi_t=L.$$

4. The distributions ψ are consistent with individual behaviour and the shock process, i.e.
$$\psi_{t+1}(B) = \int_X Q(x,t,B) d\psi_t.$$
5. The general government budget balances, i.e.
$$G = \tau(rK + wL).$$
6. The social security budget balances, i.e.
$$\theta wL = b \sum_{t=R}^{N} \mu_t$$
7. Transfers equal accidental bequests, i.e.
$$T = \left[\sum_{t=1}^{N} \mu_t (1 - s_{t+1}) \int_X a'(x,t)(1 + r(1-\tau)) d\psi_t \right] / (1+n).$$
N.B. There is a typo in the published paper.

5.1 Calibration

Calibration of the z process remains a bit controversial. Contrast Heaton and Lucas (1996) with Storesletten et al. (2001).

Let y_t be the log labour endowment and $\overline{y}_t$ is the mean for each age. Assume that the deviation from the mean satisfies the following autoregression.

$$y_t - \overline{y}_t = \gamma(y_{t-1} - \overline{y}_{t-1}) + \varepsilon_t$$

where $\varepsilon_t \sim N(0, \sigma_\varepsilon^2)$ and $y_1 \sim N(\overline{y}_1, \sigma_{y_1}^2)$.

The sequence $\overline{y}_t$ is chosen so as to match the average age–earnings profile in the data. Other parameters are chosen in a rather unsystematic way. Other studies are much better on this. See in particular Storesletten et al. (2001).

5.2 Results

See Table 4. The overall wealth Gini can be pretty well replicated. The fraction of agents with zero wealth is even over–predicted. However, the model cannot account for the high fraction of wealth held by the top percentile.

The model also has implications for what fraction of wealth has been received in the form of bequests, so–called transfer wealth. The model has plenty of transfer wealth, more than in the data. See Table 4. (The transfer wealth ratio is the ratio of transfer wealth as a fraction of total wealth.)

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